A General Design Procedure for Quarter-Wavelength Inhomogeneous Impedance Transformers Having Approximately Equal-Ripple Performance

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Abstract—A general design procedure for quarter-wavelength inhomogeneous impedance transformers having approximately equal-ripple performance is presented, based on the simplifying assumptions that the relative impedance of two waveguides of slightly different widths is a constant and that $\tan \theta_i = k_i \tan \theta_0$ in the vicinity of θ_i and $\theta_0 = 90°$. The calculation of the design parameters depends on the fact that the insertion-loss function can be expressed, in closed form, in terms of the unknown parameters. When this is identified with the permissible equal-ripple function, a set of simultaneous equations in the unknown parameters results. The solution of these equations is approximated by the solution to the corresponding homogeneous transformer problem. Thus a set of simultaneous linear equations in the small differences can be obtained which provides an approximate solution to the problem. An experimental design is described and the resulting data are presented.

Introduction

The general synthesis of impedance transformers consisting of equal-length quarter-wavelength sections of different impedance levels has been carried out by Riblet [1] and Seidel [2]. Young [3] has discussed the design problem for one- and two-section impedance transformers in which the cutoff frequencies of the transformer sections differ. He has called such impedance transformers "inhomogeneous transformers."

This paper presents a design procedure for inhomogeneous transformers involving an arbitrary number of impedance sections, subject to two simplifying assumptions which will be explained in greater detail later. This procedure employs the method of undetermined coefficients used by Collin [4] in his solution of the homogeneous transformer problem for two-, three-, and four-section transformers. On the one hand, the insertion-loss function of an arbitrary inhomogeneous transformer can readily be expressed in terms of the unknown parameters of the transformer by using the special quasi-symmetric functions[1] introduced by Riblet [1] in his discussion of the homogeneous problem. On the other hand, because of the nature of one of the approximations, the insertion-loss function of an arbitrary inhomogeneous transformer is no longer a polynomial in a suitable frequency variable. The form of these response functions is such, however, that they may be identified with equal-ripple response functions of this form, constructed by a method proposed by Riblet [5]. When the coefficients of the unknown function are identified with those of the desired function, a series of simultaneous equations determining the unknown parameters results.

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[1] These functions are defined on page 40 of reference [1]. They are used in connection with this problem and also provide a practical numerical method for solving the homogeneous problem in special cases.

The Problem

A typical inhomogeneous waveguide impedance transformer is shown in Fig. 1, together with a possible schematic. Each waveguide section is distinguished by its characteristic impedance Z_i and its cutoff wavelength λ_{ci}. This results in the fact that the λ_{gi} of the different sections will, in general, be different functions of the frequency. Now the overall transfer matrix of the cascade of transformer sections may be obtained from the product,

$$\begin{pmatrix} c_1 & js_1Z_1 \\ \dfrac{js_1}{Z_1} & c_1 \end{pmatrix} \cdot \begin{pmatrix} c_2 & js_2Z_2 \\ \dfrac{js_2}{Z_2} & c_2 \end{pmatrix} \cdots \begin{pmatrix} c_i & js_iZ_i \\ \dfrac{js_i}{Z_i} & c_i \end{pmatrix} \cdots \cdot \begin{pmatrix} c_n & js_nZ_n \\ \dfrac{js_n}{Z_n} & c_n \end{pmatrix},$$

where $c_i = \cos(\pi\bar{\lambda}_{gi}/2\lambda_{gi})$, $s_i = \sin(\pi\bar{\lambda}_{gi}/2\lambda_{gi})$, and $\bar{\lambda}_{gi}$ is the midband guide wavelength. Although the Z_i will be

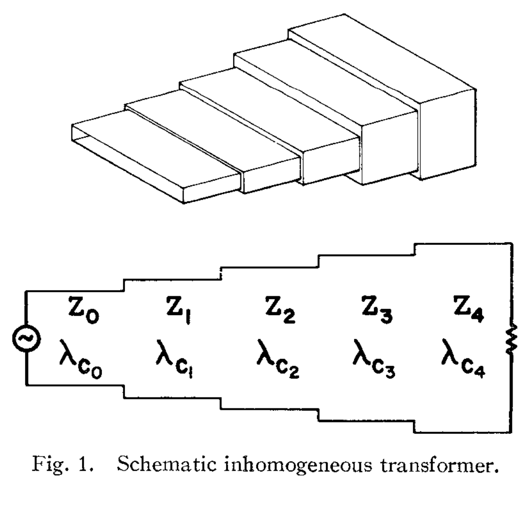

Fig. 1. Schematic inhomogeneous transformer.

treated as constant in the final design procedure, this assumption plays no part in the determination of the general insertion-loss function of the transformer.

If we define $p_i = js_i/c_i$, then the matrix product becomes

$$c_1 \cdots c_i \cdots c_n \begin{pmatrix} 1 & p_1 Z_1 \\ p_1/Z_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & p_2 Z_2 \\ p_2/Z_2 & 1 \end{pmatrix} \cdots \cdot \begin{pmatrix} 1 & p_i Z_i \\ p_i/Z_i & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & p_n Z_n \\ p_n/Z_n & 1 \end{pmatrix}.$$

When we then make the approximation that $p_i = k_i p$, where $p = js/c$ is $j \sin(\pi\bar{\lambda}_{go}/2\lambda_{go})/\cos(\pi\bar{\lambda}_{go}/2\lambda_{go})$, with λ_{go} the guide wavelength of a reference transformer section, the matrix product may be written

$$c_1/c \cdots c_n/c \begin{pmatrix} c & jk_1 s Z_1 \\ jk_1 s/Z_1 & c \end{pmatrix} \cdots \cdot \begin{pmatrix} c & jk_i s Z_i \\ jk_i s/Z_i & c \end{pmatrix} \cdots \begin{pmatrix} c & jksZ \\ jk_n s/Z_n & \end{pmatrix}.$$

On the basis of the same approximation,

$$c_i/c = 1/\sqrt{c^2 + k_i^2 s^2},$$

so that the transfer matrix of the cascade may be written

$$(c^2 + k_1^2 s^2)^{-1/2} \cdots (c^2 + k_n^2 s^2)^{-1/2} \cdot \begin{pmatrix} c & jk_1 s Z_1 \\ jk_1 s/Z_1 & c \end{pmatrix} \cdots \begin{pmatrix} c & jk_n s Z_n \\ jk_n s/Z_n & c \end{pmatrix}.$$

Here it should be observed that the determinant of each of the matrix factors when multiplied by the corresponding radical is unity as required for a lossless network.

It is seen that the foregoing transfer matrix product reduces to that for a homogeneous transformer if $k_1 = k_2, \cdots, = k_n = 1$. Moreover, if we know the input impedance or insertion-loss function of a homogeneous transformer in terms of $Z_1, \cdots, Z_n$, we can obtain the input impedance or insertion-loss function of the inhomogeneous transformer by replacing Z_i by $k_i Z_i$ and $1/Z_i$ by k_i/Z_i wherever they occur in the corresponding expression for the homogeneous transformer and then multiplying by the proper factors $(c^2 + k_i^2 s^2)^{-1/2}$.

Now the problem of finding the input impedance of a homogeneous transformer in terms of the Z_i's has been solved by Riblet [1] with the help of the elementary symmetric functions of the Z_i's. According to Riblet [1], the transfer matrix of an n section transformer is given by

$$\begin{pmatrix} c^n - \sigma_2^e s^2 c^{n-2} + \sigma_4^e s^4 c^{n-4} + \cdots & j(\sigma_1^e s c^{n-1} - \sigma_3^e s^3 c^{n-3} + \cdots) \\ j(\sigma_1^o s c^{n-3} - \sigma_3^o s^3 c^{n-3} + \cdots) & c^n - \sigma_2^o s^2 c^{n-2} + \sigma_4^o s^4 c^{n-4} + \cdots \end{pmatrix},$$

where σ_r^e is constructed by replacing each element occurring in an even-numbered position of the lexicographically ordered terms of the rth elementary symmetric function of the n variables Z_i by its inverse, while σ_r^o is obtained by the same substitution on the elements in the odd-numbered positions. Then the insertion-loss function of the network, when terminated at the input with a generator of unity impedance and at the output with a resistance of R, is given by

$$P_L = 1 + \frac{[(R-1)c^n - (R\sigma_2^e - \sigma_2^o)s^2 c^{n-2} + (R\sigma_4^e - \sigma_4^o)s^4 c^{n-4} + \cdots]^2}{4R} + \frac{[(R\sigma_1^o - \sigma_1^e)sc^{n-1} - (R\sigma_3^o - \sigma_3^e)s^3 c^{n-3} + \cdots]^2}{4R}, \tag{1}$$

while for the inhomogeneous transformer, we have

$$P_L = 1 + \frac{[(R-1)c^n - (R\bar{\sigma}_2^e - \bar{\sigma}_2^o)s^2 c^{n-2} + (R\bar{\sigma}_4^e - \bar{\sigma}_4^o)s^4 c^{n-4} + \cdots]^2}{4R(c^2 + k_1^2 s^2) \cdots (c^2 + k_n^2 s^2)} + \frac{[(R\bar{\sigma}_1^0 - \bar{\sigma}_1^e)sc^{n-1} - (R\bar{\sigma}_3^o - \bar{\sigma}_3^e)sc^{n-3} + \cdots]^2}{4R(c^2 + k_1^2 s^2) \cdots (c^2 + k_n^2 s^2)}. \tag{2}$$

Here the barred $\bar{\sigma}$'s differ from the unbarred σ's in that Z_i has been replaced by $k_i Z_i$ and $1/Z_i$ has been replaced by k_i/Z_i.

At this point, if we assume that the σ's are independent of frequency and that (2) is an exact expression for the permissible insertion-loss function, then the problem will be solved when we exhibit a function of s and c, having the form of (2), which is equal-ripple over an arbitrary frequency range and is normalized to yield the prescribed insertion loss at zero frequency. Of course, these assumptions are not exact. For this reason, the design obtained in this way is said to be "approximately equal-ripple."

For ideal equal-ripple performance, it is readily argued that the last term in the preceding expression for P_L is identically zero. To see this, divide numerator and denominator of the fractions by s^{2n}. Then with $\omega = c/s$,

$$P_L = 1 + \frac{P_n^2(\omega) + Q_{n-1}^2(\omega)}{(k_1^2+\omega^2)(k_2^2+\omega^2)\cdots(k_n^2+\omega^2)},$$

where $P_n(\omega)$ is a polynomial in ω of degree n, and $Q_{n-1}(\omega)$ is a polynomial in ω of degree $n-1$. Riblet [5] has pointed out a number of procedures for determining the optimum equal-ripple response of this form. For it,

$$P_n^2(\omega) + Q_{n-1}^2(\omega) = E_n^2(\omega),$$

where $E_n^2(\omega)$ has n double zeros in the pass band. Since P_n and Q_{n-1} have real coefficients, they must each have simple zeros at the double zeros of $E_n^2(\omega)$; but $Q_{n-1}(\omega)$ cannot have n simple zeros without vanishing identically. Thus, as a general consequence of equal-ripple performance, we require that

$$R\bar{\sigma}^0_{2k+1} - \bar{\sigma}^e_{2k+1} = 0, \quad k = 0, \cdots, n/2 - 1. \tag{3}$$

Now $E_n(\omega)$ can be expressed as a homogeneous polynomial in s and c divided by s^n. That is,

$$E_n(\omega) = E_n(s, c)/s^n,$$

where

$$E_n(s, c) = \gamma_n c^n - \gamma_{n-2}s^2c^{n-2} + \gamma_{n-4}s^4c^{n-4} + \cdots.$$

Comparing this with the second term of (2), we see that $\gamma_n = (R-1)/2\sqrt{R}$, and that

$$R\bar{\sigma}_{2k}^e - \bar{\sigma}_{2k}^0 = 2\sqrt{R}\gamma_{n-2k}, \quad k = 1, \cdots, n/2. \tag{4}$$

Equations (3) and (4) constitute a system of nonlinear equations in the k's and Z's since R and the γ's are specified by the design problem. They constitute, in some sense, a formal solution of the problem.

As will be seen in the example to be considered in the next section, there are more unknown Z's and k's than there are defining equations. Accordingly, the solution of an inhomogeneous transformer design problem in general contains a high degree of indeterminancy. The additional degrees of freedom may be used to determine "optimum" equal-ripple designs. For example, a shortest design might be the objective as Young [3] has suggested with a two-section transformer. For most engineering purposes, however, this ambiguity can be avoided by assigning a waveguide height to each waveguide width since the k's depend only on the width of the transformer section under study, while the Z's involve both the width and height of the transformer section. Of course, the selected correspondence must be consistent with the cross sections of the terminating waveguides. Moreover, if it is smoothly monotonic, the error of many of the approximations used will be minimized. In the example developed later, a linear relationship between height and width is employed.

For small values of n, the solution of (3) and (4) can be determined without great difficulty if it is kept in mind that an approximate solution is available. If we assume a smoothly monotonic transformer, the inhomogeneous solution does not differ too much from the homogeneous solution. To each impedance value of the homogeneous solution there corresponds a definite value of k. Thus (3) and (4) may be converted into conditions on the differences between the Z's of the homogeneous solution and their corresponding k's and the exact Z's and k's. Because these differences are small, the resulting conditions are linear in these differences and the differences may be determined numerically.

Equations (3) and (4) are useful in connection with the design of homogeneous transformers as well. The conditions on the impedances of an equal-ripple homogeneous transformer are given by (3) and (4) when the $\bar{\sigma}$'s are replaced by the σ's and the γ's are properly constructed from the coefficients of the Chebyshev polynomials. For values of n up to 6 at least, an approximate solution of a homogeneous transformer problem, as provided by tables calculated by Young [6], can readily be improved upon by using the method of small differences.

It will be useful, in the numerical example to be considered later, and possibly helpful here to exhibit explicit expressions for the σ's and for (3) and (4), when $n=3$. Then

$$\begin{aligned}
\sigma_1^o &= 1/Z_1 + 1/Z_2 + 1/Z_3; & \sigma_1^e &= Z_1 + Z_2 + Z_3 \\
\sigma_2^o &= Z_2/Z_1 + Z_3/Z_1 + Z_3/Z_2; & \sigma_2^e &= Z_1/Z_2 + Z_1/Z_3 \\
& & &\quad + Z_2/Z_3 \\
\sigma_3^o &= Z_2/Z_1Z_3; & \sigma_3^e &= Z_1Z_3/Z_2
\end{aligned} \tag{5}$$

while

$$\begin{aligned}
\bar{\sigma}_1^o &= k_1/Z_1 + k_2/Z_2 + k_3/Z_3; & \bar{\sigma}_1^e &= k_1Z_1 + k_2Z_2 + k_3Z_3 \\
\bar{\sigma}_2^o &= k_1k_2Z_2/Z_1 + k_1k_3Z_3/Z_1 & \bar{\sigma}_2^e &= k_1k_2Z_1/Z_2 \\
&\quad + k_2k_3Z_3/Z_2; & &\quad + k_1k_3Z_1/Z_3 \\
& & &\quad + k_2k_3Z_2/Z_3 \\
\bar{\sigma}_3^o &= k_1k_2k_3Z_2/Z_1Z_3 & \bar{\sigma}_3^e &= k_1k_2k_3Z_1Z_3/Z_2.
\end{aligned} \tag{6}$$

Then (3) and (4) are

$$\begin{aligned}
&k_1(R/Z_1 - Z_1) + k_2(R/Z_2 - Z_2) + k_3(R/Z_3 - Z_3) = 0 \\
&\qquad RZ_2/Z_1Z_3 - Z_1Z_3/Z_2 = 0 \\
&k_1k_2(RZ_1/Z_2 - Z_2/Z_1) + k_1k_3(RZ_1/Z_3 - Z_3/Z_1) \\
&\qquad + k_2k_3(RZ_2/Z_3 - Z_3/Z_2) = 2\sqrt{R}\,\gamma_1.
\end{aligned} \tag{7}$$

For $k_1 = k_2 = k_3 = 1$, these are just the equations for the homogeneous, three-section transformer. Then the first two equations are satisfied by $Z_1Z_3 = R$ and $Z_2^2 = R$, while the third equation for suitable γ_1 reduces to (23) as given by Collin [4].

The determination of the required equal-ripple functions involves expressions with denominators containing factors of the form $\sqrt{k_i^2 + (1-k_i^2)x^2}$, if we put $c^2 + s^2 = 1$. In order that the approximation $\tan\theta_i = k_i \tan\theta_0$ hold as well as possible over the range of guide-widths for a given frequency band, it is desirable to select the guide width used in defining θ_0 as a mean between the guide widths at the two ends of the transformer. Thus k_i will vary from values less than one to values greater than one. Then the roots of $k_i^2+(1-k_i^2)x^2$ are partially on the real axis and partially on the imaginary axis. This complicates the problem of applying the primitive equal-ripple functions proposed by Riblet [5] since the k's are as yet unknown. For this reason, it is convenient to construct a set of primitive equal-ripple functions having denominators of the desired form. Consider the angle δ defined by the equation

$$e^{i\delta} = \frac{(L+1)e^{i\phi} - (L-1)e^{-i\phi}}{2\sqrt{L^2 + (1-L^2)x^2}} \tag{8}$$

where $x = \cos\phi$ over the range $0 < \phi < \pi$. By the same methods used by Riblet [5] it is readily argued that δ increases monotonically from 0 to π, except possibly for integral multiples of 2π, as ϕ increases from 0 to π for all real values of L. The primitive equal-ripple function with the same behavior over the limited range $|x| \le \mu \le 1$ is obtained by replacing x by x/μ and renormalizing, with the result that in

$$e^{i\delta} = \frac{\{k + \sqrt{k^2+\mu^2(1-k^2)}\}e^{i\phi} - \{k - \sqrt{k^2+\mu^2(1-k^2)}\}e^{-i\phi}}{2\sqrt{k^2+(1-k^2)x^2}} \tag{9}$$

where $x = \mu\cos\phi$, δ increases monotonically from 0 to π as ϕ increases from 0 to π for all real values of k. Of course, $k=1$ is a possible value, but this situation causes no difficulty in the behavior of δ. The optimum equal-ripple functions in the variable c are then constructed by evaluating expressions of the form

$$\mathrm{Re}\ \{e^{i\delta_1}\cdot e^{i\delta_2}\cdots e^{i\delta_n}\}. \tag{10}$$

Experimental and Numerical Example

By way of illustration, we undertook the design of a three-section transformer from a 1.590 by 0.795-inch waveguide to a 1.372 by 0.200 inch waveguide to cover a 500 Mc/s frequency band centered at 6.175 Gc/s.[2] The determination of the impedance transformation R is the first problem. If $\bar{\lambda}_{gi}$ is then to be the midband guide wavelength of the ith transformer section, it is given by the familiar expression

$$2\lambda_{gi}^u\lambda_{gi}^l/(\lambda_{gi}^u + \lambda_{gi}^l), \tag{11}$$

where λ_{gi}^u and λ_{gi}^l are the guide wavelengths of the ith transformer section at the upper and lower limits of the frequency band, respectively. If $\bar{\lambda}_i$ is then defined as the corresponding value of the free-space wavelength, it will be found that $\bar{\lambda}_i = -1.9124$ for the 1.372-inch waveguide and $\lambda_i = -1.9120$ for the 1.590-inch waveguide. Thus we can use 1.912 inches as the midband wavelength of all sections of the transformer with negligible error. As the formula for the characteristic impedance of waveguide, we now use the simple rule "height times guide wavelength," for reasons given in the Appendix. Expressed in symbols,

$$Z = b\lambda_g. \tag{12}$$

In evaluating this expression for the two terminating waveguides[3] at their common midband frequency, it is found that $R=3.566$ if the characteristic impedance of the smaller waveguide is normalized to unity.

Now the equal-ripple response is given by

$$P_L = 1 + \frac{(R-1)^2}{4R}\left\{\frac{E_3(s,c)}{E_3(0,1)}\right\}^2. \tag{13}$$

Moreover, $E_3(s,c)$ is obtained by expression $\mathcal{E}_3(c)$ in a form homogeneous in s and c with the help of the identity $s^2+c^2=1$, and $E_3(c)$ is given by

$$E_3(c) = \mathrm{Re}\left\{\frac{[(k_1+\sqrt{1})e^{i\phi} - (k_1-\sqrt{1})e^{-i\phi}]\cdots[(k_3+\sqrt{3})e^{i\phi} - (k_3-\sqrt{3})e^{-i\phi}]}{8\sqrt{k_1^2+(1-k_1^2)c^2}\sqrt{k_2^2+(1-k_2^2)c^2}\sqrt{k_3^2+(1-k_3^2)c^2}}\right\}. \tag{14}$$

Here $\sqrt{n} = \sqrt{k_n^2+\mu^2(1-k_n^2)}$. When this multiplication is carried out and the real part is evaluated, it is found that

$$E_3(c) = \frac{(k_1k_3\sqrt{2} + k_1k_2\sqrt{3} + k_2k_3\sqrt{1} + \sqrt{1}\sqrt{2}\sqrt{3})c^3/\mu^3 - (k_1k_3\sqrt{2} + k_1k_2\sqrt{3} + k_2k_3\sqrt{1})c/\mu}{\sqrt{k_1^2+(1-k_1^2)c^2}\sqrt{k_2^2+(1-k_2^2)c^2}\sqrt{k_3^2+(1-k_3^2)c^2}}. \tag{15}$$

[2] This problem arose from a commercial requirement. Undoubtedly, better examples can be found to illustrate the limitations of the theory.

[3] Strictly speaking, (12) is justified only for small changes in cross section. In the design of an impedance transformer, however, we are concerned only with the interaction of the reflections established at each step. If the steps are all small, then (12) is applicable at each step and is thus the formula to be applied to the terminating cross sections in the design procedure, even though it will not give the overall reflection at a direct junction of these cross sections.

Of course,

$$E_3(1) = \frac{(k_1k_3\sqrt{2} + k_1k_2\sqrt{3} + k_2k_3\sqrt{1} + \sqrt{1}\sqrt{2}\sqrt{3}) - (k_1k_3\sqrt{2} + k_1k_2\sqrt{3} + k_2k_3\sqrt{1})\mu^2}{\mu^3}. \tag{16}$$

Then,

$$\frac{E_3(s, c)}{E_3(0, 1)} = \frac{c^3 - \dfrac{(k_1k_2\sqrt{3} + k_1k_3\sqrt{2} + k_2k_3\sqrt{1})\mu^2 s^2 c}{k_1k_2\sqrt{3} + k_1k_3\sqrt{2} + k_2k_3\sqrt{1} + \sqrt{1}\sqrt{2}\sqrt{3} - (k_1k_2\sqrt{3} + k_1k_3\sqrt{2} + k_2k_3\sqrt{1})\mu^2}}{\sqrt{k_1^2s^2 + c^2}\sqrt{k_2^2s^2 + c^2}\sqrt{k_3^2s^2 + c^2}}. \tag{17}$$

Thus

$$\gamma_1 = \frac{R - 1}{2\sqrt{R}} \cdot \frac{\mu^2}{(1 - \mu^2) + \dfrac{\sqrt{1}\cdot\sqrt{2}\cdot\sqrt{3}}{k_1k_2\sqrt{3} + k_1k_3\sqrt{2} + k_2k_3\sqrt{1}}}. \tag{18}$$

For values of k not differing too greatly from 1 and bandwidths less than one octave, or any combination of these conditions,

$$\frac{\sqrt{1}\sqrt{2}\sqrt{3}}{k_1k_2\sqrt{3} + k_1k_3\sqrt{2} + k_2k_3\sqrt{1}} \approx 1/3 + \frac{\mu^2}{3}(k_1^{-2} + k_2^{-2} + k_3^{-2} - 3). \tag{19}$$

Thus the equal-ripple condition for a three-section inhomogeneous transformer may be written

$$\begin{aligned} &k_1k_2(RZ_1/Z_2 - Z_2/Z_1) + k_1k_3(RZ_1/Z_3 - Z_3/Z_1) \\ &\quad + k_2k_3(RZ_2/Z_3 - Z_3/Z_2) \\ &= (R - 1)\mu^2 \Big/ \left[4/3 - \mu^2 + \frac{\mu^2}{3} \cdot (k_1^{-2} + k_2^{-2} + k_3^{-2} - 3)\right]. \end{aligned} \tag{20}$$

This reduces to (23) of Collin [4] for $k_1 = k_2 = k_3 = 1$ since his $\cos\theta_2 = \sqrt{3}\mu/2$. If the basic frequency variable of the problem is associated with a guide width in between 1.590 and 1.372 inches, then the coefficient of $\mu^2/3$ will be small. Thus this term can be neglected and the k's occur only on the left-hand side of the equation. Essentially, we are arguing at this point that the constants on the right side of (4) are the same as those to be used in solving the homogeneous transformer problem for the same R and bandwidth and the guide width used in defining θ_0.

Now k_i is defined to give an approximate solution to the equation $\cot\theta_0 = k_i \cot\theta_i$, where $\theta_0 = \pi\bar{\lambda}_g/2\lambda_g$ is the the electrical length of some mean waveguide and $\theta_i = \pi\bar{\lambda}_{gi}/2\lambda_{gi}$ is the electrical length of the ith transformer section. If k_i is selected so that $\cot\theta_0$ and $k_i \cot\theta_i$ have the same slope at midband, then it is found that k_i is proportional to $\bar{\lambda}_{gi}^{-2}$. To determine the mean transformer width, we require that the values of k at the input and output terminals be equidistant from unity. Then

$$\frac{\bar{\lambda}_{go}^2}{\bar{\lambda}_{gn}^2} - 1 = 1 - \frac{\bar{\lambda}_{go}^2}{\bar{\lambda}_{gw}^2}, \tag{21}$$

where $\bar{\lambda}_{go}$ is the guide wavelength of the mean-width transformer at midband, while $\bar{\lambda}_{gn}$ is the midband wavelength at the narrow end of the transformer and $\bar{\lambda}_{gw}$ is the midband guide wavelength at the wide end of the transformer. From (21), we have

$$\bar{\lambda}_{go}^2 = 2\bar{\lambda}_{gn}^2 \cdot \bar{\lambda}_{gw}^2/(\bar{\lambda}_{gn}^2 + \bar{\lambda}_{gw}^2). \tag{22}$$

For the example under consideration, $\bar{\lambda}_{gn} = 2.668$ and $\bar{\lambda}_{gw} = 2.393$. Then $\bar{\lambda}_{go} = 2.519$, which in turn corresponds to a waveguide width of 1.468 inches. This then is the mean transformer width and uniquely defines the basic frequency variable $\cot\theta_0$.

A solution of (20) requires the values of μ. Now $\mu = |\cos\theta_0|$ at the edges of the design band. It is readily determined that $\mu = 0.110$. To solve (7) for the equal-ripple case as restricted by (20), we avail ourselves of the fact that both Z and k for a given transformer section depend only on the width of the waveguide, since some expression for the height of each transformer section in terms of its width has already been agreed upon. Thus k is a function of Z given implicitly by the conditions already noted. If the characterisitic impedances Z_1, Z_2 and Z_3 of the three sections of the transformer are known to have the approximate values Z_1', Z_2', and Z_3', we may write

$$Z_1 = Z_1' + dZ_1$$
$$Z_2 = Z_2' + dZ_2$$
$$Z_3 = Z_3' + dZ_3$$

and

$$k_1 = k_1' + (dk/dZ)_1 dZ_1$$
$$k_2 = k_2' + (dk/dZ)_2 dZ_2$$
$$k_3 = k_3' + (dk/dZ)_3 dZ_3$$

where k' is the value of k corresponding to the approximate value of Z, Z', and dk/dZ is evaluated at the three approximate values of Z. When these are substituted in (7) and higher powers of the dZ's are neglected, a set of simultaneous linear equations in the three dZ's result. These are then readily solved.

For the example considered the Z's were selected to be solutions of the homogeneous equal-ripple transformer for $\mu = 0.11$ and $R = 3.566$. Then $Z_1' = 1.175$, $Z_2' = 1.888$, and $Z_3' = 3.034$. When the values of the dZ's are calculated, the final solution is $Z_1 = 1.155$, $Z_2 = 1.818$, and $Z_3 = 2.972$. To these impedance values correspond waveguide widths of 1.384, 1.437, and 1.537 inches and waveguide heights of 0.233, 0.379, and 0.649 inch. When the quarter wavelengths were corrected according to Cohn [7], the final design shown in Fig. 2 resulted. Here, in the application of Cohn's formulas for the change in length of the quarter-wave sections due to the susceptance of the steps, the susceptance B, due to the change in height given by the curves on page 309 of reference [8], was simply placed in parallel with the shunt reactance X [8, page 300].

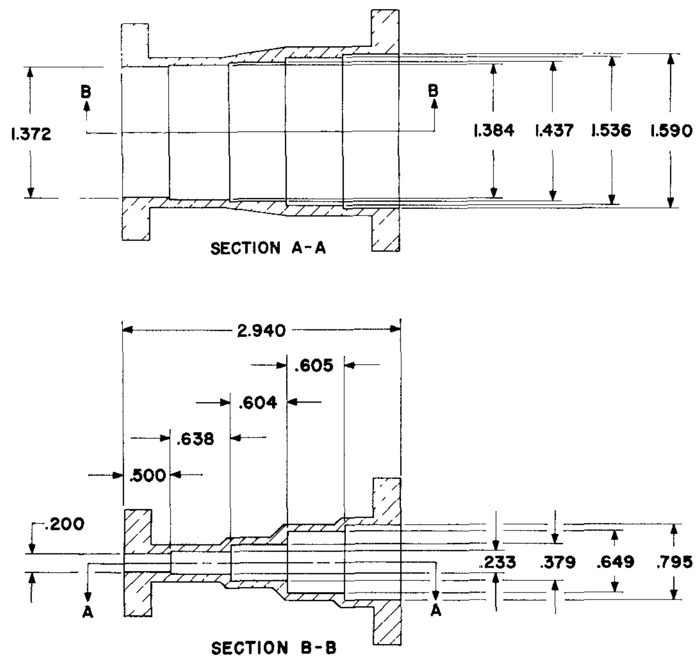

Fig. 2. Three-section transformer, experimental design (WR-159 to WR-137 by 0.200 height).

It will be seen that the reflection coefficients at the four steps of the approximate solution are 1.175, 1.607, 1.607, and 1.175, respectively, compared with the corrected values of 1.155, 1.574, 1.635, and 1.120. The average correction is only about 0.025 so that the effort of this type of calculation is warranted only when a low VSWR is required. For the example considered, the reflection coefficient of the entire transformer over the design band was specified to be less than 0.015. It would clearly require an elaborate analysis to prove that errors of 0.025 at each of four steps are negligible under this condition.

It is useful to derive a formula for the maximum VSWR in the pass band of this transformer. At the edge of the pass band where $c = \mu$, $E_3(\mu) = \pm 1$. Then by (13)

$$P_L(\mu) = 1 + \frac{(R-1)^2}{4R} E_3(0, 1)^{-2}. \tag{23}$$

Now

$$E_3(0, 1) \approx [4 + \mu^2(k_1^{-2} + k_2^{-2} + k_3^{-3} - 3)]k_1k_2k_3/\mu^3; \tag{24}$$

and then

$$P_L(\mu) \approx 1 + \frac{(R-1)^2}{4R} \cdot \left\{\frac{\mu^3}{4k_1k_2k_3}\right\}^2. \tag{25}$$

This reduces to the homogeneous case, for μ small, if $k_1 = k_2 = k_3 = 1$. In our numerical example, $k_1k_2k_3 \approx 1$ by our choice of the frequency variable. Hence the VSWR of an inhomogeneous transformer in its pass band will approximate that of a homogeneous transformer between the same impedance values if a waveguide width is used which is a mean of the two terminating waveguides. For the case in question, $R = 3.566$, $\mu = 0.11$, and the approximate pass band VSWR is 1.0004.

Validity of the Approximations

We can put an upper limit on the error in the approximation $\cot\theta_0 = k_i \cot\theta_i$ by applying it to the terminating waveguides. This is done in Fig. 3 where $\cot\theta_0$ is plotted from 4.9 to 8.2 kMc/s and compared with $k_n \cot\theta_n$ for the narrow terminating waveguide and with $k_w \cot\theta_w$ for the wide terminating waveguide. Over the frequency band which is common to the recommended frequency bands of the two waveguide sizes, the error due to the approximation is somewhat less than 1 percent. Even over the frequency band which includes both recommended bands, there is a maximum error of about 8 percent at the low frequency extreme. Thus it is felt that this approximation may be used safely in the design of most inhomogeneous transformers.

Somewhat more difficult is the estimation of the errors resulting from the fact that the Z's are not constant but vary with frequency according to their guide wavelength. Strictly speaking, then, (3) and (4) hold only at the midband frequency. This means that the coefficients of the insertion-loss function P_L in (2) vary somewhat about their ideal midband values, with the result that the response is no longer truly equal-ripple. When these coefficients were evaluated as a function of frequency, it was found that a certain amount of compensation was present so that the coefficients themselves were less sensitive to frequency than the impedance values comprising them. This is probably related to the fact that, in a monotonic solution, the reflection coefficients at each junction vary in the same way with frequency, while for the first-order solution of a homogeneous transformer problem only the ratio of reflection coefficients is determined by the requirement of equal-

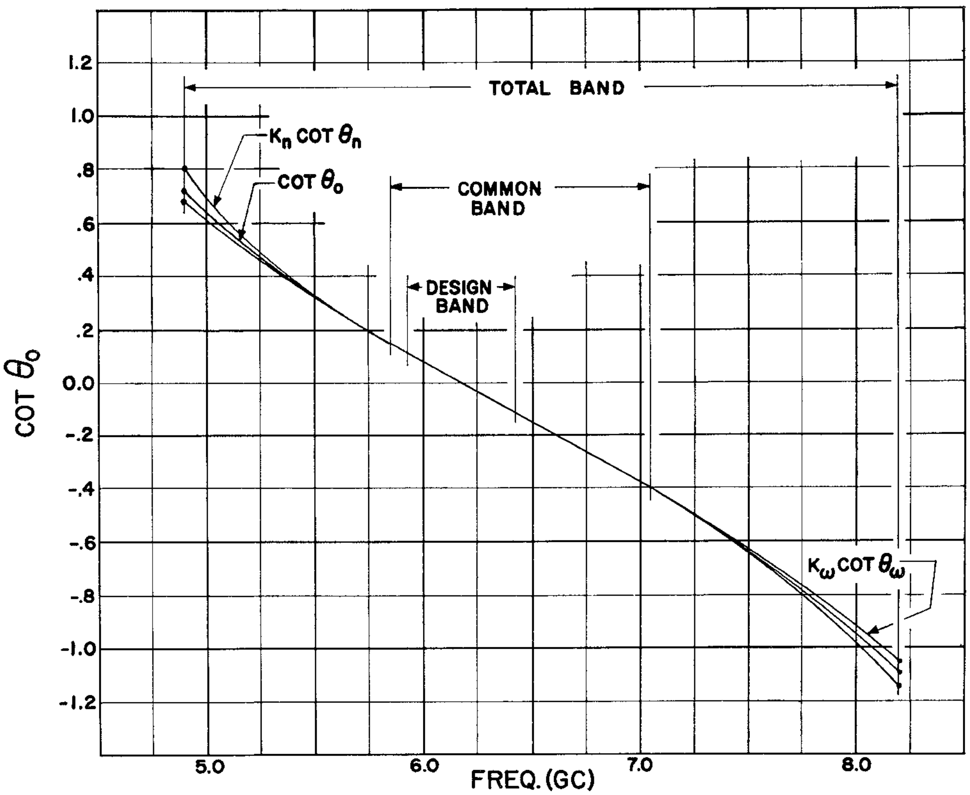

Fig. 3. Range of approximation, cot $\theta_0 = k_n$ cot θ_n.

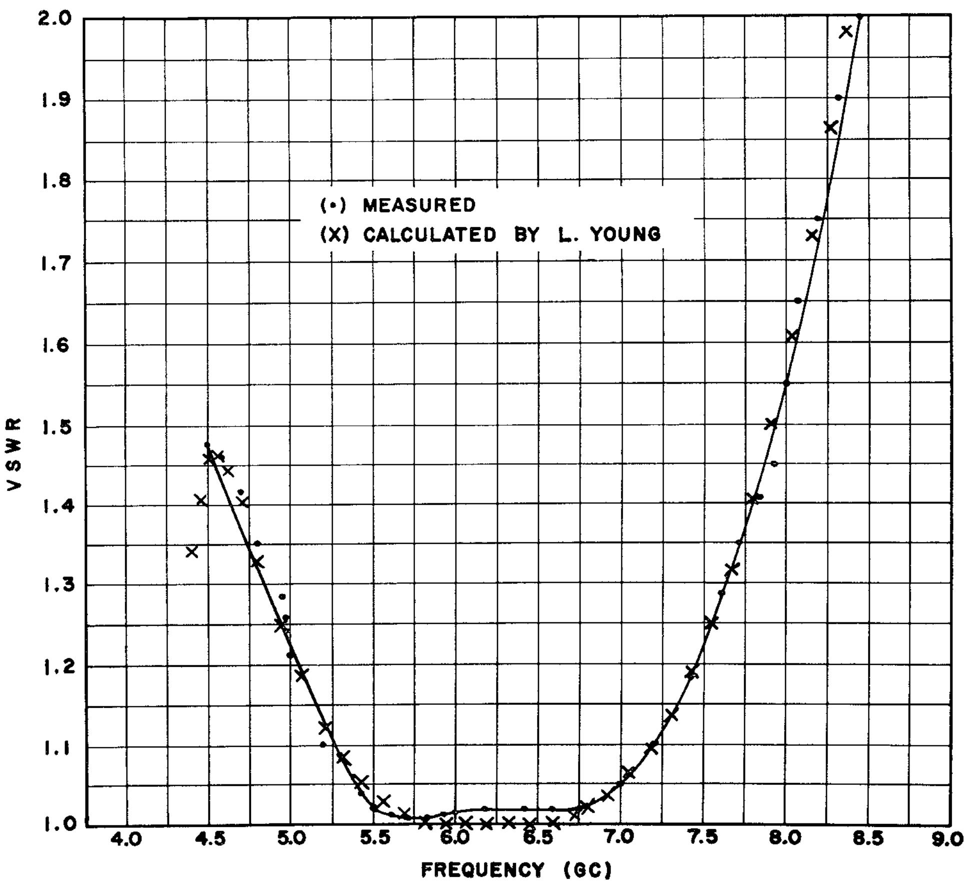

Fig. 4. Performance of experimental design.

ripple performance. This error is thus probably related to the change in pass-band VSWR due to the change in R as a function of frequency.

Experimental Confirmation

Figure 4 gives the measured VSWR of an inhomogeneous transformer which was carefully electroformed in accordance with the dimensions of Fig. 2. The band-broadening is unexpected, but more or less consistent with the deterioration in the pass band of the VSWR tolerance. This in turn is not surprising in view of the approximations in the theory and the various experimental errors. However, the VSWR is less than 1.03 over the design band and so falls within our original design specification. It is interesting to note that the line-length corrections used gave a frequency response which was well centered about the design band. It will be observed also that the data terminate abruptly on the low-frequency side. This arises from the fact that we were operating so close to the cutoff frequency of the narrow waveguide that accurate measurements on the low-frequency side proved to be too costly.

The writer is indebted to Dr. Leo Young for the computations, also shown in Fig. 4, which were made on the basis of a prepublication version of this paper. Taking a program that he had used in connection with his earlier interest in the design of inhomogeneous transformers, Young calculated the exact response of the transformer—using, however, the formula $b\lambda_g/a\lambda$ for the frequency dependence of the characteristic waveguide impedance of the transformer sections. Two salient facts stand out from these calculations. First, the observed data of Fig. 4 and Young's calculations agree surprisingly well at the skirts of the transformer response. Second, at the center of the band, Young's calculations show a suggestion of equal-ripple performance over substantially the desired band with a maximum VSWR less than 1.01, even though the use of formula $b\lambda_g/a\lambda$ could have been expected to introduce some additional error. Thus one may conclude that the measured deterioration of the pass-band VSWR to 1.03 is primarily due to experimental error, while the band-broadening at the skirts is due to errors in the approximations.

Conclusion

A general design procedure is presented for waveguide transformers which makes allowance for the fact that each of the transformer sections may have a different cutoff frequency. An experimental model is designed, and its performance is given. It is found that a slight change in impedance values is all that is required to compensate for a substantial change in waveguide width.

Appendix

The formula used for waveguide impedance $b\lambda_g$ is simply a product of formulas for the impedance change, for change in height, multiplied by a formula for the impedance change, for change in width. It is well known and amply verified by experiment that the VSWR due to a small change in waveguide height can be accurately predicted by assuming that the impedance of rectangular waveguide is proportional to its height. It is not so well known that the VSWR due to small changes in width can be accurately predicted by assuming that the impedance of rectangular waveguide is proportional to its guide wavelength.

This latter fact follows from (1c) of Marcuvitz [8]. If his primed and unprimed notation is replaced by subscripts n and w for narrow and wide waveguides, then this equation may be written,

$$\frac{Z_n}{Z_w} \approx \frac{\lambda_{gn} a_n}{\lambda_{gw} a_w} (1 + \beta) \qquad \text{for } \beta \ll 1;$$

but $\beta = 1 - \alpha = 1 - a_n/a_w$, where $a_n/a_w \approx 1$. Thus $1+\beta = 2 - a_n/a_w \approx a_w/a_n$. So that finally

$$Z_n/Z_w \approx \lambda_{gn}/\lambda_{gw}.$$

Thus the impedance change is proportional to the guide wavelength.

To test the composite formula $b\lambda_g$ for waveguide impedance, two experimental transitions were measured. In the first, the input waveguide was 0.900 by 0.400 inch while the output waveguide was 0.843 by 0.286 inch. In the second, the input waveguide was 0.900 by 0.200 inch while the output waveguide was 0.970 by 0.348 inch. In both experiments, the measured VSWR agreed with the calculated VSWR within the experimental error over the frequency band 8.2 to 12.4 Gc/s.

References

[1] H. J. Riblet, "General synthesis of quarter-wave impedance transformers," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-5, pp. 36–43, January 1957.

[2] H. Seidel, "Synthesis of a class of microwave filters," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-5, pp. 107–114, April 1957.

[3] L. Young, "Inhomogeneous quarter-wave transformers of two sections, *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-8, pp. 645–649, November 1960.

[4] R. E. Collin, "Theory and design of wide-band multisection quarter-wave transformers," *Proc. IRE*, vol. 43, pp. 179–185, February 1955.

[5] H. J. Riblet, "The application of a new class of equal-ripple functions to some similar transmission-line problems," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-12, pp. 415–421, July 1964.

[6] L. Young, "Stepped-impedance transformers and filter prototypes," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-10, pp. 339–359, September 1962.

[7] S. B. Cohn, "Optimum design of stepped transmission-line transformers," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-3, pp. 16–21, April 1955.

[8] N. Marcuvitz, *Waveguide Handbook*, vol. 10, MIT Radiation Lab. Ser. New York: McGraw-Hill, 1951.